Status of the Hadronic τ Decay Determination of $|V_{us}|$

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We update the hadronic τ determination of $|V_{us}|$, showing that current strange branching fractions produce results $2-3\sigma$ lower than 3-family unitarity expectations. Issues related to the size of theoretical uncertainties and results from an alternate, mixed τ-electroproduction sum rule determination are also considered.

1. Introduction and Background

The determination of $|V_{us}|$ from hadronic τ decay data rests on the finite energy sum rule (FESR) relation,

$$\int_0^{s_0} w(s)\,\rho(s)\,ds \;=\; -\frac{1}{2\pi i}\oint_{|s|=s_0} w(s)\,\Pi(s)\,ds \quad (1)$$

valid for any analytic $w(s)$ and kinematic-singularity-free correlator, Π, having spectral function, $\rho(s)$. To obtain $|V_{us}|$, Eq. (1) is applied to the flavor-breaking (FB) correlator difference $\Delta\Pi_\tau(s) \equiv \left[\Pi^{(0+1)}_{V+A;ud}(s) - \Pi^{(0+1)}_{V+A;us}(s)\right]$, where $\Pi^{(J)}_{V/A;ij}$ are the spin $J=0,1$ components of the flavor ij, vector (V) or axial vector (A) current two-point functions, and $(0+1)$ denotes the sum of $J=0$ and 1 components. The OPE is to be employed on the RHS for sufficiently large s_0.

The spectral functions, $\rho^{(J)}_{V/A;ij}$, are related to the differential distributions, $dR_{V/A;ij}/ds$, of the normalized flavor ij V or A current induced decay widths, $R_{V/A;ij} \equiv \Gamma[\tau^- \to \nu_\tau\,\mathrm{hadrons}_{V/A;ij}\,(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]$, by [1]

$$\frac{dR_{V/A;ij}}{ds} = c_\tau^{EW}|V_{ij}|^2\left[w^{(00)}_{L+T}(y_\tau)\rho^{(0+1)}_{V/A;ij}(s) - w^{(00)}_L(y_\tau)\rho^{(0)}_{V/A;ij}(s)\right] \quad (2)$$

with $y_\tau = s/m_\tau^2$, $w^{(00)}_{L+T}(y) = (1-y)^2(1+2y)$, $w^{(00)}_L(y) = 2y(1-y)^2$, V_{ij} the flavor ij CKM matrix element, and, with S_{EW} a short-distance electroweak correction [2], $c_\tau^{EW} \equiv 12\pi^2 S_{EW}/m_\tau^2$.

Use of the $J=0+1$, FB difference $\Delta\Pi_\tau$, rather than the analogous difference involving the linear combination of $J=0,0+1$ spectral functions appearing in Eq. (2), is a consequence of the extremely bad behavior of the integrated $J=0$ (longitudinal) $D=2$ OPE series [3]. Fortunately, apart from the accurately known π and K pole terms, contributions to $\rho^{(0)}_{V+A;ij}$ are $\propto [(m_i \mp m_j)^2]$, making ud continuum contributions negligible. Once the small continuum us $J=0$ contributions are determined phenomenologically using dispersive [4] and sum rule [5] analyses of the strange scalar and pseudoscalar channels, respectively, the $J=0$ contributions can be subtracted, bin-by-bin, from $dR_{V+A;ij}/ds$, allowing one to construct the re-weighted $J=0+1$ spectral integrals, $R^w_{V+A;ij}(s_0)$, defined by

$$\frac{R^w_{V+A;ij}(s_0)}{c_\tau^{EW}|V_{ij}|^2} \equiv \int_0^{s_0} ds\, w(s)\,\rho^{(0+1)}_{V+A;ij}(s)\ , \quad (3)$$

and, from these, the FB combinations,

$$\delta R^w_{V+A}(s_0) = \frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{V+A;us}(s_0)}{|V_{us}|^2} = c_\tau^{EW}\int_0^{s_0} ds\, w(s)\Delta\rho_\tau(s)\ . \quad (4)$$

With $|V_{ud}|$, and any parameters in $\delta R^{w,OPE}_{V+A}(s_0) = c_\tau^{EW}\left[\frac{-1}{2\pi i}\oint_{|s|=s_0} ds\, w(s)\Delta\Pi_\tau(s)\right]$

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from other sources, Eq. (1) yields [6]

$$|V_{us}| = \sqrt{\frac{R^w_{V+A;us}(s_0)}{\frac{R^w_{V+A;ud}(s_0)}{|V_{ud}|^2} - \delta R^{w,OPE}_{V+A}(s_0)}} . \qquad (5)$$

$\delta R^{w,OPE}_{V+A}(s_0)$ is typically $<<$ $R^w_{V+A;ud,us}(s_0)$ (usually at the few-to-several-% level) for $s_0 \gtrsim 2\ \mathrm{GeV}^2$, making a high precision $|V_{us}|$ determination possible with only modest OPE precision [6].

It turns out (see also below) that the convergence of the integrated $J = 0+1$, $D = 2$ OPE series may also be somewhat problematic. As a result, it is also of interest to consider FESRs based on the alternate FB correlator difference,

$$\Delta\Pi_M \equiv 9\Pi_{EM} - 5\Pi^{(0+1)}_{V;ud} + \Pi^{(0+1)}_{A;ud} - \Pi^{(0+1)}_{V+A;us}, \quad (6)$$

where Π_{EM} is the scalar part of the electromagnetic (EM) current two-point function. $\Delta\Pi_M$ shares with $\Delta\Pi_\tau$ the vanishing of $D = 0$ contributions to all orders but, by construction, has strongly suppressed $D = 2$ contributions [7]. $D = 4$ contributions turn out also strongly suppressed compared to those of $\Delta\Pi_\tau$. This suppression does not, however, persist beyond $D = 4$ [7]. The EM spectral function, $\rho_{EM}(s)$, required on the LHS of the $\Delta\Pi_M$ FESR, is given by $\rho_{EM}(s) = \frac{s\sigma_0(s)}{16\pi^2\alpha^2_{EM}}$, with $\sigma_0(s)$ the bare inclusive hadronic electroproduction cross-section. The $\Delta\Pi_M$ FESR yields a solution for $|V_{us}|$ of the form Eq. (5), with the RHS denominator replaced by $9R^w_{EM}(s_0) - 5\frac{R^w_{V;ud}(s_0)}{|V_{ud}|^2} + \frac{R^w_{A;ud}(s_0)}{|V_{ud}|^2} - \delta R^{OPE}_{w,M}(s_0)$, where $R^w_{EM}(s_0) = c^{EW}_\tau \int_0^{s_0} ds\, w(s)\rho_{EM}(s)$ and

$$\delta R^{OPE}_{w,M}(s_0) = \frac{-c^{EW}_\tau}{2\pi i} \oint_{|s|=s_0} ds\, w(s)\Delta\Pi_M(s) \, . \quad (7)$$

2. Spectral and OPE Input

2.1. Spectral Input

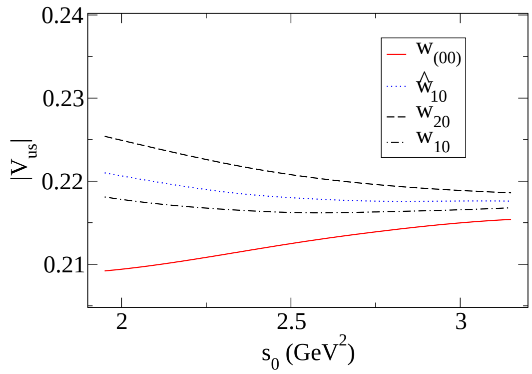
We compute $R^w_{V+A;ud}(s_0)$ and $R^w_{V+A;us}(s_0)$ using the publicly available ALEPH us [8] and ud [9] spectral data and covariances. Separate ud V and A analogues, $R^w_{V/A;ud}(s_0)$, required for the mixed τ-electroproduction FESRs implement the improved $\bar{K}K\pi$ V/A ud separation [10] provided by CVC and the BaBar determination of $I = 1$ $K\bar{K}\pi$ electroproduction cross-sections [11]. A small global rescaling of the continuum ud V+A distribution accounts for recent changes in S_{EW}, $R_{V+A;us}$ and B_e. We employ $|V_{ud}| = 0.97425(23)$ [12] and current values [13] for B_e, $R_{V+A;us}$ and $R_{V+A;ud}$. Since BaBar and Belle have not yet completed their remeasurements of $dR_{V+A;us}/ds$, we work with an interim partial update obtained by rescaling the 1999 ALEPH distribution [8] mode-by-mode by the ratio of new to old world averages for the branching fractions [14]. The new world averages, based on the results of Refs. [15,16,17,18,19,20,21,22], are given in Table 1 [13]. The us V+A covariance matrix cannot yet be analogously updated, so the improved precision on the us branching fractions translates into an improved us spectral integral error only for $w = w^{(00)}_{L+T} \equiv w_{(00)}$ and $s_0 = m_\tau^2$.

Details of the treatment of the EM spectral data, required for the spectral integral side of the $\Delta\Pi_M$ FESR, are omitted here because of space constraints, but may be found in Ref. [7].

Table 1
Current world averages, $B[\tau^- \to X^-_{us}\nu_\tau]$ for the main strange hadronic modes X^-_{us}, with S factors where required. Column 3 gives the references associated with post-PDG2006 changes.

X^-_{us}	$\mathcal{B}_{WA,2008}$ (%)	Refs.
K^- [τ decay]	0.690(10)	[13,18]
([$K_{\mu 2}$])	(0.715(4))	
$K^-\pi^0$	0.426(16)	[20]
$\bar{K}^0\pi^-$	0.835(22) ($S = 1.4$)	[17,22]
$K^-\pi^0\pi^0$	0.058(24)	
$\bar{K}^0\pi^0\pi^-$	0.360(40)	
$K^-\pi^-\pi^+$	0.290(18) ($S = 2.3$)	[16,21]
$K^-\eta$	0.016(1)	[22]
$(\bar{K}3\pi)^-$ (est'd)	0.074(30)	
$K^-\omega$	0.067(21)	
$(\bar{K}4\pi)^-$ (est'd)	0.011(7)	
$K^{*-}\eta$	0.014(1)	[22]
$K^-\phi$	0.0037(3) ($S = 1.3$)	[16,19]
TOTAL	2.845(69)	
	(2.870(68))	

Figure 1. $|V_{us}|$ versus s_0 from the $\Delta\Pi_\tau$ FESRs for, from top to bottom, w_{20}, $\hat{w}_{10}$, w_{10} and $w_{(00)}$.



2.2. OPE input

To keep OPE-breaking contributions from the vicinity of the timelike point on $|s| = s_0$ sufficiently suppressed, we restrict our attention to $w(s)$ having at least a double zero at $s = s_0$, and to $s_0 \gtrsim 2$ GeV^2 [23].

The leading, $D = 2$, OPE contribution to $\Delta\Pi_\tau$ is known to 4 loops [24]:

$$\left[\Delta\Pi_\tau(Q^2)\right]^{OPE}_{D=2} = \frac{3}{2\pi^2}\frac{m_s(Q^2)}{Q^2}\left[1 + \frac{7}{3}\bar{a} + 19.93\bar{a}^2 + 208.75\bar{a}^3 + \cdots\right] , \qquad (8)$$

with $\bar{a} = \alpha_s(Q^2)/\pi$, and $\alpha_s(Q^2)$ and $m_s(Q^2)$ the running coupling and strange quark mass in the $\overline{MS}$ scheme. Since independent determinations of α_s imply $\bar{a}(m_\tau^2) \simeq 0.1$, convergence at the spacelike point on $|s| = s_0$ is marginal at best. With such slow convergence, conventional prescriptions for assessing the $D = 2$ truncation uncertainty may lead to significant underestimates.

To deal with the potential $D = 2$ convergence problem, one may either work with $\Delta\Pi_\tau$ and $w(s)$ chosen to emphasize regions of the complex $s = -Q^2$-plane away from the spacelike point, where $|\alpha_s(Q^2)|$ is smaller and convergence improved [25], or switch to the alternate $\Delta\Pi_M$ FESRs where $D = 2$ contributions are suppressed already at the correlator level [7]. In the latter case, the $D = 2$ contribution becomes

$$\left[\Delta\Pi_{\tau,EM}(Q^2)\right]^{OPE}_{D=2} = \frac{3}{2\pi^2}\frac{m_s(Q^2)}{Q^2}\left[\frac{1}{3}\bar{a} + 4.384\bar{a}^2 + 44.94\bar{a}^3 + \cdots\right] , \qquad (9)$$

more than an order of magnitude smaller than in the $\Delta\Pi_\tau$ case. Since $\alpha_s(s_0)$ grows with decreasing s_0, making higher order terms relatively more important at lower scales, extracted $|V_{us}|$ results will display an unphysical s_0-dependence if neglected, higher order $D = 2$ terms are, in fact, not negligible. s_0-stability studies thus provide a handle on the impact of the potentially slow integrated $D = 2$ convergence on $|V_{us}|$.

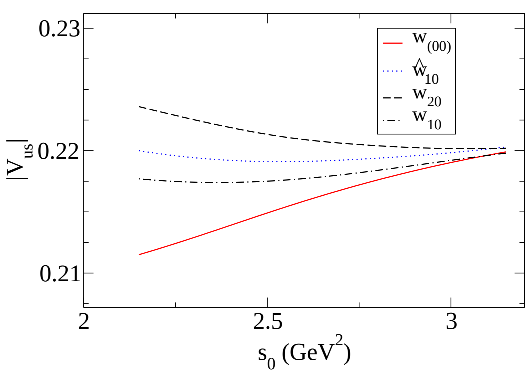
$D = 4$ OPE contributions to $\Delta\Pi_\tau(Q^2)$ and $\Delta\Pi_M(Q^2)$ are determined by $\langle m_s\bar{s}s\rangle$ and $\langle m_\ell\bar{\ell}\ell\rangle$, up to negligible $O(m_s^4)$ corrections. The relevant expressions, as well as those for the $D = 6$ four-quark condensate contributions, are easily constructed from the results of Ref. [26], and given in Ref. [7]. If one works with weights $w(s) = \sum_{m=0} b_m y^m$, with $y = s/s_0$, integrated $D = 2k + 2$ OPE terms scale as $1/s_0^k$, allowing contributions of different D to be distinguished by their differing s_0-dependences.

As $D = 2$ OPE input, we employ $m_s(2\ \mathrm{GeV}) = 96(10)$ MeV [27] and $\alpha_s(m_\tau^2) = 0.323(9)$, the latter obtained from an average, $\alpha_s(M_Z) = 0.1190(10)$, of various recent determinations (including lattice [28] and τ [29] results, which are now in very good agreement) via the standard combination of 4-loop running and 3-loop matching at the flavor thresholds [30].

At $D = 4$, we employ the GMOR relation for $\langle m_\ell\bar{\ell}\ell\rangle$ and evaluate $\langle m_s\bar{s}s\rangle$ using the ChPT determination of m_s/m_ℓ [31] and $\langle m_s\bar{s}s\rangle/\langle m_\ell\bar{\ell}\ell\rangle = 1.2(3)$, the latter obtained by updating Ref. [32] using the average of recent $n_f = 2+1$ lattice determinations of f_{B_s}/f_B as input [33].

$D = 6$ contributions are estimated using the vacuum saturation approximation (VSA), rescaled by $\rho_{VSA} = 1(5)$, while $D > 6$ contributions are neglected. Since integrated $D \geq 6$ OPE contributions scale as $1/s_0^N$ ($N \geq 2$), if $D > 4$

Figure 2. $|V_{us}|$ versus s_0 from the $\Delta\Pi_\tau$ FESRs for, from top to bottom, w_{20}, $\hat{w}_{10}$, w_{10} and $w_{(00)}$, with the spectral input modified by rescaling up by 3σ the branching fraction of the large, but not yet remeasured, $\bar{K}^0\pi^-\pi^0$ mode.

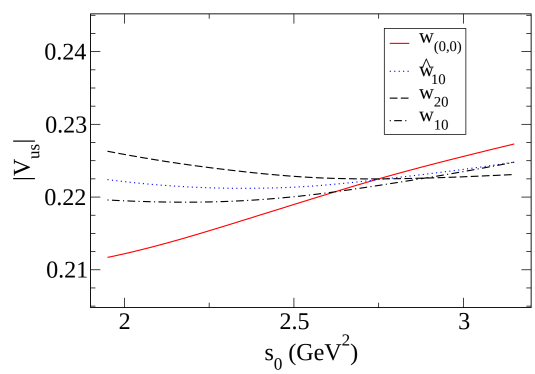


contributions are, in fact, not small, and these input assumptions are unreliable, an unphysical s_0-dependence of $|V_{us}|$ will result, again making s_0-stability tests important.

3. Results and discussion

The results for $|V_{us}|$ obtained using the inputs specified above for the $\Delta\Pi_\tau$ FESRs based on the $J = 0+1$ kinematic weight $w_{(00)}(y)$, and three weights, $w_{10}(y)$, $w_{20}(y)$, $\hat{w}_{10}(y)$, constructed in Ref. [25] specifically to improve the poor integrated $J = 0+1$, $D = 2$ convergence, are displayed in Fig. 1. The s_0-instability of the $w_{(00)}$ results is much greater than the theoretical uncertainty $\sim \pm 0.0005$ often quoted for the $s_0 = m_\tau^2$ version of this analysis in the literature. The results corresponding to $\hat{w}_{10}$, in contrast, display a very good window of s_0-stability. A positive feature of the $\Delta\Pi_\tau$ analysis is the fact that the results for all four weights appear to be converg-

Figure 3. $|V_{us}|$ versus s_0 from the $\Delta\Pi_\tau$ FESRs for, from top to bottom, w_{20}, $\hat{w}_{10}$, w_{10} and $w_{(00)}$, with the spectral input modified by rescaling up by 3σ the branching fractions of all modes not yet remeasured by either BaBar or Belle.



ing towards the stable $\hat{w}_{10}$ value as $s_0 \to m_\tau^2$. The $s_0 = m_\tau^2$ versions of the various analyses are

$$|V_{us}| = \begin{cases} 0.2180(32)(15) & (\hat{w}_{10}) \\ 0.2188(29)(22) & (w_{20}) \\ 0.2172(34)(11) & (w_{10}) \\ 0.2160(26)(8) & (w_{(00)}) \end{cases} \qquad (10)$$

where the first error is experimental (dominated by us spectral errors) and the second the nominal theoretical error. The nominal theory error is obviously much smaller than the observed s_0-instability in the $w_{(00)}$ case, and hence unrealistically small. Comparison to the results of earlier $\Delta\Pi_\tau$ FESR analyses [6,27,34,35] shows the significant impact of recent, improved us experimental results on the $|V_{us}|$ central values. The decreases represented by the remeasured us branching fractions, lead to $|V_{us}|$ results $2-3\sigma$ below the 3-family-unitarity expectation, 0.2255(1) [12].

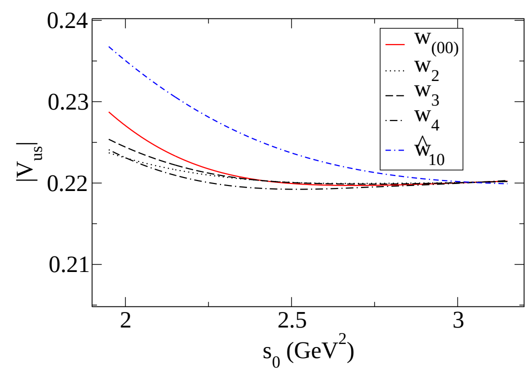
It should be stressed that several important strange decay modes have yet to be remeasured

by either BaBar or Belle, and that the level of consistency of the $s_0 = m_\tau^2$ results for different weights could be significantly affected by such future remeasurements. As an illustration, we show, in Figure 2, the impact on $|V_{us}|$ as a function of s_0 of rescaling upward by 3σ the as-yet-unremeasured $\bar{K}^0\pi^-\pi^0$ branching fraction, and hence also the $\bar{K}^0\pi^-\pi^0$ component of the us spectral distribution employed above. The issue of whether plausible shifts in the as-yet-unremeasured branching fractions are capable of restoring agreement with 3-family-unitarity expectations is less clear. In fact, it would take simultaneous 3σ upward rescalings of all currently unremeasured us branching fractions to restore agreement. Such a rescaling, moreover, does not produce a convincing s_0-stability plateau for any of the weights considered, as shown in Figure 3.

The results for $|V_{us}|$ for the $\Delta\Pi_M$ FESRs based on $w_{(00)}(y)$, the weight $\hat{w}_{10}(y)$ displaying the best s_0-stability for the $\Delta\Pi_\tau$ FESR, and the weights w_2, w_3 and w_4, where $w_N = 1 - \frac{N}{N-1}y + \frac{y^N}{N-1}$, are displayed in Fig. 4. The weight w_N produces a single surviving integrated $D = 2N+2 > 4$ OPE contribution suppressed by the coefficient $1/(N-1)$ and scaling as $1/s_0^N$, making it a useful choice in this case, where the slow integrated $D = 2$ convergence found for the w_N versions of the $\Delta\Pi_\tau$ FESRs is no longer relevant.

If it was poor $D = 2$ convergence which was responsible for the s_0-instability of the $w_{(00)}$ $\Delta\Pi_\tau$ FESR results, one would expect to see a much improved stability plateau for the corresponding $\Delta\Pi_M$ FESR, as is indeed found. The very good stability for the w_N results also indicates that the integrated $D = 2N+2$ contributions relevant to these cases become negligible in the upper part of the s_0 window displayed in the Figure. Since, however, $D \geq 6$ contributions increase in going from $\Delta\Pi_\tau$ to $\Delta\Pi_M$ [7], one would expect the instability for weights like $\hat{w}_{10}$, which do not suppress these to the same extent as do the other weights considered, to be enhanced, as is indeed found to be the case. Even so, the $\hat{w}_{10}$ results converge well to the stable results for the other weights as $s_0 \to m_\tau^2$.

Figure 4. $|V_{us}|$ as a function of s_0 from the mixed τ-electroproduction FESRs for, from top to bottom at the left, $\hat{w}_{10}$, $w_{(00)}$, w_3, w_4 and w_2.



Given the very good stability of the $w_{(00)}$ results, it is possible to quote a final result based on the $s_0 = m_\tau^2$ version of the $w_{(00)}$ FESR, which allows us to take advantage of the improvements in the us branching fraction errors. The result is

$$|V_{us}| = 0.2208(27)(28)(5)(2) \qquad (11)$$

where the first three errors are due to the uncertainties on the us V+A, residual $I = 0$ EM and residual ud V/A spectral integrals, respectively, and the fourth is due to the $D = 2$ and 4 OPE uncertainties (see Ref. [7] for further details).

We conclude by stressing that, for both the $\Delta\Pi_\tau$ and $\Delta\Pi_M$ FESRs, improved errors on $dR_{V+A;us}/ds$ are crucial. This requires both remeasurements of as-yet-unremeasured strange mode branching fractions, pursuit of higher multiplicity modes with branching fractions down to the few$\times 10^{-5}$ level, and, in particular, a full investigation of the $K3\pi$ and $K4\pi$ modes, which were not in fact measured, but rather estimated, in the earlier experimental analyses.

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